The mass formula for quasi-black holes

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A quasi-black hole, either non-extremal or extremal, can be broadly defined as the limiting configuration of a body when its boundary approaches the body's quasihorizon. We consider the mass contributions and the mass formula for a static quasi-black hole. The analysis involves careful scrutiny of the surface stresses when the limiting configuration is reached. It is shown that there exists a strict correspondence between the mass formulas for quasi-black holes and pure black holes. This perfect parallelism exists in spite of the difference in derivation and meaning of the formulas in both cases. For extremal quasi-black holes the finite surface stresses give zero contribution to the total mass. This leads to a very special version of Abraham-Lorentz electron in general relativity in which the total mass has pure electromagnetic origin in spite of the presence of bare stresses.

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I. INTRODUCTION

When the size of a compact body approaches its own gravitational radius, usually the pressure, naturally assumed finite, cannot support the body itself anymore, and gravitational

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collapse starts with some margin before the gravitational radius is attained. Concurrently, it turns out that there exist systems which possess static configurations as close to the gravitational radius as one likes (see [1] and references therein). They are called quasi-black holes and they possess a would-be horizon, called a quasihorizon, instead of an event horizon as for black holes. A quasi-black hole represents a particular kind of a black hole mimicker, configurations close to be black holes but having no event horizon [2]. Typical properties of quasi-black holes consist in that the values of the lapse function on the boundary and everywhere inside it tend to zero, giving rise to whole regions of infinitely large redshifts. Such systems can be found in quite different contexts, namely, self-gravitating monopoles, extremal charged dust, either compact or dispersed [3, 4], extremal charged shells [5], and shells gluing Reissner-Nordström and Bertotti-Robinson spacetimes [6, 7] (see also [2]). Quasi-black holes, with finite stresses, should be extremal, where for extremal one means that the mass M of such objects equals the charge Q, $M = Q$ [1]. For a static object, this typically requires electric charge, or some other form of appropriate repulsive charge.

Now, we want to extend the definition of quasi-black holes. In [1], we indeed proved a theorem showing that quasi-black holes should be extremal. This theorem was proved under the restrictive assumption that the stresses on the surface of the body, the surface stresses, on the quasihorizon should remain finite. Now, we want to drop this assumption and allow for unbounded surface stresses on the quasihorizon. Without restricting the surface stresses to be bounded on the quasihorizon, there are certainly many more different types of configurations that can achieve a quasihorizon. One can have now, also non-extremal objects, as well as the extremal ones of the previous considerations [1]. So, a quasi-black hole, can be broadly defined as the limiting configuration of a body, either non-extremal or extremal, when its boundary approaches the body's quasihorizon. We will also include in our discussion ultraextremal quasi-black holes, where the metric functions have a special behavior, related to the extremal case but distinct [8, 9, 10]. This ultraextremal behavior also arises within black hole or cosmological solutions. For instance, in the Reissner-Nordström-de Sitter solution, an ultraextremal triple horizon forms due to the existence of a special relation between mass, charge and cosmological constant.

This enlargement in the definition of quasi-black holes will prove crucial in the derivation of a quite general mass formula for the quasi-black holes themselves, the main aim of this work. Indeed, our analysis shows that there is a non-trivial connection between two pairs

of two different issues: (i) between surface stresses and the mass formula for quasi-black holes, and (ii) between the mass formula for quasi-black holes and the well known mass formula for black holes. Concerning point (i) the ability to allow for, not only finite, but also infinite stresses enlarges the spectrum of objects and gives rise naturally to a mass formula. Concerning (ii) we show the close correspondence between the mass formulas for quasi-black holes and pure black holes both in the non-extremal and extremal cases. We want to emphasize that both, the physical nature of the objects and the derivation of the mass formula, is quite different for quasi-black holes and black holes in turn. This certainly makes the close relationship between the mass formulas non-trivial. Our analysis has also rather unexpected consequences for the general relativistic counterpart of the classical model of the Abraham-Lorentz electron. These features are related with the distinguished role played by the quasihorizon in the extremal case.

II. METRIC FORM AND EXTENSION OF THE NOTION OF STATIC QUASI-BLACK HOLES TO ENCOMPASS NON-EXTREMAL CASES

A. Metric form and definition of a quasi-black hole embodying the non-extremal case

Let us have a distribution of matter in a gravitational field which does not depend on time. Put the four-dimensional spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, with μ, ν being spacetime indices, in the form

$$ds^2 = -N^2 dt^2 + g_{ik}\left(dx^i + N^i dt\right)\left(dx^k + N^k dt\right) , \tag{1}$$

where, we use 0 as a time index, and $i, k = 1, 2, 3$ as spatial indices. In addition, N and N^i are the lapse function and shift vector which depend in general on the spatial coordinates x^i. Putting $N^i = 0$ to study the static case, the metric (1) reduces to

$$ds^2 = -N^2 dt^2 + g_{ik}dx^i dx^k , \tag{2}$$

where N is a function of the spatial coordinates. It is further convenient to work in Gauss normal coordinates where the metric looks like (see [11], and e.g., [12])

$$ds^2 = -N^2 dt^2 + dl^2 + g_{ab}dx^a dx^b , \tag{3}$$

with l being a radial coordinate, and a, b representing the other two spatial coordinates. For instance, if the metric is spherically symmetric they are the angular coordinates θ and ϕ. Note that Gaussian normal coordinates cannot be extended beyond the point where geodesics normal to the surface begin to form caustics. However, we are interested in the vicinity of the body's surface only, which is going to become a quasihorizon, so for our purposes it is quite sufficient to use the reference system (3) in this vicinity.

We suppose that the body is compact with its boundary approaching the quasihorizon [3] or, as in [4], the distribution can be dispersed but with a well-defined quasi-black hole limit. We consider the static case, and assume no further symmetry, spherical or whatever. Previously, the definition of a quasi-black hole, and the corresponding quasihorizon, was done in [1] for spherically symmetric spacetimes. Its generalization to static spacetimes without the requirement of spherical symmetry leads to the following points. Consider a configuration depending on a parameter ε such that (a) for small but non-zero values of ε the metric is regular everywhere with a non-vanishing lapse function N, at most the metric contains only delta-like shells, (b) taking as ε the maximum value of the lapse function on the boundary $N_{\rm B}$, then in the limit $\varepsilon \to 0$ one has that the lapse function $N \leq N_{\rm B} \to 0$ everywhere in the inner region, (c) the Kretschmann scalar Kr remains finite in the quasihorizon limit. This latter property implies another important property which can be stated specifically, namely, (d) the area A of the two-dimensional boundary $l = \text{const}$ attains a minimum in the limit under consideration, i.e., $\lim_{\varepsilon\to 0} \frac{\partial A}{\partial l}|_{l^*} = 0$, where l^* is the value of l at the quasihorizon. When a configuration obeys these three properties (a)-(c) (or (a)-(d)) we say one is in the presence of a quasi-black hole, enlarging to non-spherically symmetric spacetimes, though still static, the definition given in [1]. A remark should be made: In many cases of physical interest, especially for the spherically symmetric systems, the lapse function is a monotonically decreasing function of the proper distance in the direction from the boundary to the inner region (see Appendix B in [1]). Then, we can weaken point (b) and require only that the maximum boundary value obeys $N_{\rm B} \to 0$.

B. Finiteness of the Kretschmann scalar: Elaboration of property (c)

Let us elaborate on property (c). In [1] only extremal configurations were considered. As discussed in [1], these are regular in the sense that the Kretschmann scalar Kr remains

finite in the quasihorizon limit (as property (c) above demands), and in addition the surface stresses (if any) also remain regular in that limit. Now, for the non-extremal case the attempt to use the same notion for a quasi-black hole leads to infinite stresses as shown in [1], so that one should allow for infinite stresses if one wants to include the non-extremal case. Therefore one should also ask whether or not for non-extremal quasi-black holes one should insist, as in (c) above, that Kr remains finite. We showed in a previous article [2] that, typically, point (c) can be violated for mimickers, generic configurations close to be black holes but having no event horizon, of which a quasi-black hole is an example. For such singular configurations the notion of a quasi-black hole, non-extremal one, would be devoid of meaning. Therefore, to keep non-extremal quasi-black holes as physically relevant objects, we demand that point (c) should be maintained in the list of properties above as an important requirement.

1. Non-extremal case

We will now see what consequences property (c) by looking at the explicit expression for Kr. We will follow [11] closely, where true black holes, rather than quasi-black holes, were considered. One can obtain from (3) that the Kretschmann scalar Kr is given by

$$\mathrm{Kr} = P_{ijkl}P^{ijkl} + 4C_{ij}C^{ij} \,, \tag{4}$$

where P_{ijkl} is the curvature tensor for the subspace $t = \text{const}$, and

$$C_{ij} = \frac{N_{|ij}}{N} \,, \tag{5}$$

with $_{|i}$ denoting the covariant derivative with respect to the corresponding three-dimensional metric. As the metric of the three-space is positive definite, all terms enter the entire expression (4) with a positive sign, so that each term should be finite separately.

For instance, let us see the pure black hole case as done in [11]. The finiteness of Kr entails that in the horizon limit, when $N \to 0$, the numerator in C_{ij} (see Eq. (5)) must vanish. Without loss of generality, for the non-extremal horizon one can choose $l = 0$. Considering then different combinations of indices, one arrives at the asymptotic form of N for the pure black hole case [11],

$$N = \kappa l + \kappa_3 \frac{l^3}{3!} + O(l^4) \tag{6}$$

where $\kappa = \text{const}$ is the surface gravity of the black hole, and κ_3 is some function.

Now, we want to study the quasi-black hole case instead of the black hole case. Choosing the coordinate l in such a way that $l = 0$ on the boundary surface, where $N = N_0(x^a) \to 0$, in the limit under discussion we obtain that

$$\lim_{l\to 0} C_{ll} = \frac{\lim_{l\to 0} N''}{N_0} \tag{7}$$

where $' \equiv \frac{\partial}{\partial l}$, and,

$$\lim_{l\to 0} C_{al} = \frac{\lim_{l\to 0} N'_{;a}}{N_0}\,, \tag{8}$$

where $_{;a}$ means the covariant derivative with respect to the metric g_{ab} in Eq. (3). We can write $N_0 = \varepsilon f(x^a)$, with $\varepsilon = 0$ corresponding to the quasi-black hole limit. Then, it follows from the finiteness of Kr that

$$\lim_{\varepsilon\to 0}\lim_{l\to 0} N'' = 0\,, \tag{9}$$

$$\lim_{\varepsilon\to 0}\lim_{l\to 0} N'_{;a} = 0\,. \tag{10}$$

If we write the expansion for small l in the form

$$N = N_0 + \kappa_1(x^a,\varepsilon)l + \kappa_2(x^a,\varepsilon)\frac{l^2}{2!} + +\kappa_3(x^a,\varepsilon)\frac{l^3}{3!} + O(l^4)\,, \tag{11}$$

and take into account (9)-(10) we obtain that

$$\lim_{\varepsilon\to 0} \kappa_1(x^a,\varepsilon) = \kappa \tag{12}$$

is a constant, and

$$\lim_{\varepsilon\to 0} \kappa_2 = 0. \tag{13}$$

Thus, we see that the expansion (11) has the same structure as (6). From the meaning of a quasi-black hole, we want in the outer region to have, $\lim_{\varepsilon\to 0} N(\varepsilon; l, x^a) = N_{\rm bh}(l, x^a)$, where $N_{\rm bh}$ is the lapse function for a black hole (this can be not the case for the inner region, see [1]). Now, in (12) we wrote the limit is κ, but strictly speaking we should have put $\kappa_{\rm h}$, the surface gravity of the quasi-black hole. Therefore, the final step consists in identifying indeed $\kappa_{\rm h}$ with the surface gravity κ, so that we a have a well-defined limit for the quantity κ_1. Thus, as far as the properties of the metric are concerned, we have proved that

$$\lim_{l\to 0}\lim_{\varepsilon\to 0} = \lim_{\varepsilon\to 0}\lim_{l\to 0}\,. \tag{14}$$

For a configuration which is close to the quasi-black hole limit but does not attain it, there are slight deviations of the coefficients κ_1 and κ_2 from their limiting values but, the closer to the limit is the configuration, the smaller the corrections become. In the quasi-black hole limit one can ignore those corrections altogether and consider, in particular, the surface gravity as a constant, similarly to what happens to black holes. So, $\kappa_{\mathrm{h}} = \kappa$. We would like to stress that the validity of the expansion (11) with the additional properties (12)-(13) is an essential property. For example, in [2] we considered black hole mimickers for which $N = \sqrt{V + \varepsilon^2}$, so that $\lim_{\varepsilon\to 0}\lim_{l\to 0}\frac{\partial^2 N}{\partial l^2}$ diverges and the expansion (11) fails. Such configurations have singular limits, indeed Kr diverges. Thus, not any deformation of a black hole metric depending on some deformation parameter ε (see [2]) is suitable if we want to give a reasonable extension of the notion of quasi-black black holes to the non-extremal case. The requirement (c) formulated above selects then admissible deformations among the possible ones.

We show now how property (d) follows from the above requirements. Similarly to the expansion for the lapse function N (see Eq. (11)), we can write the expansion for the metric g_{ab} as,

$$g_{ab} = \overset{(0)}{g_{ab}}(x^a) + \overset{(1)}{g_{ab}}(x^a) l + \frac{\overset{(2)}{g_{ab}}}{2}(x^a) l^2 + O(l^3)\,. \tag{15}$$

Then, from the requirement of the finiteness of C_{ab} we obtain for the extrinsic curvature, K_{ab}, define here as $K_{ab} = -\frac{1}{2}\frac{\partial g_{ab}}{\partial n}$, that

$$\lim_{\varepsilon\to 0}\lim_{l\to 0} K_{ab} = 0\,, \tag{16}$$

similarly to the property $\lim_{l\to 0}\lim_{\varepsilon\to 0} K_{ab} = 0$ which is known to hold for black holes [11]. Finally, for quasi-black holes, we obtain that the area of the cross-section $l = \mathrm{const}$ obeys,

$$\lim_{\varepsilon\to 0}\lim_{l\to 0}\frac{1}{A}\frac{\partial A}{\partial l} = -\frac{1}{2}\lim_{\varepsilon\to 0}\lim_{l\to 0} K_{ab} = 0\,, \tag{17}$$

which is just property (d) mentioned at the end of Sec. II A.

2. *Extremal and ultraextremal cases*

In the extremal case the situation is even simpler. Consider a quasi-black hole in which a small parameter, $\varepsilon \neq 0$ say, enumerates configurations. Then, by the definition of a quasi-black hole, $\lim_{\varepsilon\to 0} N(l^*, x^a; \varepsilon) = 0$, where $l^* = l^*(\varepsilon)$ corresponds to the proper distance

between any fixed point and the quasihorizon [1]. Here one comment is in order. Actually, in relation to the non-extremal case, we use a somewhat different definition of proper distance here. This simply reflects the qualitatively different properties of the non-extremal and extremal black hole geometries to which a corresponding quasi-black hole tends in the outer region, in the limit $\varepsilon \to 0$. Indeed, for a non-extremal black hole the proper distance from the horizon to any other point is finite, so that without loss of generality we have adapted l so that $l = l^* = 0$ for the quasihorizon itself. In the extremal case the proper distance from the horizon to any other point is infinite, so one has to measure l not from the horizon but from any other fixed point, $l \to \infty$, when the second point approaches the horizon. Correspondingly, for a quasi-black hole in the extremal case the proper distance from a fixed point to a quasi-horizon $l^*(\varepsilon)$ is finite but $\lim_{\varepsilon\to 0} l^*(\varepsilon) = \infty$, justifying thus our choice for the extremal case. Now, we take into account that if a continuous function, $f(x)$ say, of an arbitrary variable x, is such that the limit $f_\infty = \lim_{x\to\infty} f(x)$ exists and is the finite (roughly speaking, the function approaches asymptotically a constant value), it follows that $\lim_{x\to\infty} \frac{df}{dx} = 0$. So, this means that $\lim_{\varepsilon\to 0} \frac{\partial N}{\partial l^*} = 0$. It can be rewritten as $\lim_{\varepsilon\to 0} \left(\frac{\partial N}{\partial l}\right)_{\mathrm{h}} = 0$, where the subscript h means here that the quantity is calculated on the quasihorizon. If we take the limits in the other order, we simply return to the usual black hole, which, by definition, in the limit $l \to \infty$ the lapse function behaves as $N \sim \exp(-Bl)$ with $B = \mathrm{const} > 0$ in the extremal case. For an ultraextremal quasi-black hole, one has that the asymptotic behavior of the lapse function is given by $N \sim l^{-n}$, with $n > 0$, and the choice of l^* is the same as for the extremal case. There are also black holes that have this ultraextremal behavior, for example, the Reissner-Nordström-de Sitter solution with a triple horizon occuring due to a special relation between mass, charge and cosmological constant [8, 9, 10]. The ultraextremal quasi-black hole and black hole cases have the same correspondence between themselves as the extremal cases. Thus, in brief, the property that extremal and ultraextremal black holes have zero surface gravity at their horizon has a corresponding identical property for extremal and ultraextremal quasi-black holes at their quasihorizon.

We show now, for the extremal and ultraextremal cases, how property (d) follows from the above requirements. The metric g_{ab} in the extremal case, say, has the following expansion,

$$g_{ab} = g_{ab}^{(0)} + g_{ab}^{(1)} \exp(-\frac{l}{l_0}) + g_{ab}^{(1)} \exp(-\frac{2l}{l_0}) + ... \tag{18}$$

where l_0 is a constant. The maximum value of l is equal to l^*, which corresponds to the value of the proper distance between a fixed point and the quasi-horizon. For a finite parameter ε the quantity l^* is also finite, but $l^* \to \infty$ when $\varepsilon \to 0$. Then, again, we obtain that the tensor K_{ab} on the quasi-horizon, in the limit $\varepsilon \to 0$, obeys $K_{ab} \to 0$. Correspondingly, property (d) holds (note here that in Eq. (17) in the limiting process, the value $l = 0$ corresponding to the quasi-horizon should be replaced by $l = \infty$). In the ultraextremal case we have, instead of (18), an expansion with respect to inverse powers of l that also leads to Eq. (17), i.e., to property (d).

3. *The three cases together: Non-extremal, extremal and ultraextremal*

Now, for extremal and ultra extremal cases, the surface gravity obeys $\kappa = 0$ by definition. So, we can combine all three cases, i.e., non-extremal, extremal and ultraextremal, in the formula,

$$\lim_{\varepsilon \to 0} \left(\frac{\partial N}{\partial l} \right)_{\mathrm{h}} = \kappa \,, \tag{19}$$

where again the subscript h means here that the quantity is calculated on the quasihorizon, and where κ is equal to the surface gravity of the corresponding black hole. Roughly speaking, for the outer region, a quasi-black hole represents an object that realizes the limiting transition from an "would-be black hole" to a true one. Therefore, it is not surprising that there is a direct correspondence between their features.

Let us conclude this section with some general remarks. Actually, the properties (a)-(c) listed in Sec. II A mean that in the limit $\varepsilon \to 0$ the metric of a quasi-black hole approaches that of a black hole everywhere in the outer region (let us stress again that this is not necessarily the case for the inner region because of the complex entanglement between coordinates and parameters in the course of the limiting process [1]). Therefore, it is quite trivial that far from the quasi-horizon the derivatives of the metric also coincide in the limit under discussion. However, in the vicinity of the quasi-horizon, because of the interplay between two small parameters, ε and l in the non-extremal case, or ε and $1/l$ in the extremal or ultraextremal cases, this is not obvious in advance, and additional substantiation is necessary to establish Eq. (19), as was done in the above consideration. In addition, in all three cases the property (d) holds.

III. THE MASS FORMULA FOR THE GENERIC STATIC CASE

If the matter is joined onto a vacuum spacetime then one has to be careful and use the junction condition formalism [13, 14]. The mass of the matter distribution can be written as an integral over the region occupied by matter and fields. Defining T_μ^ν as the stress-energy tensor, the mass is given by the Tolman formula (see, e.g., [15], or for the original work [16], see also [17]),

$$M = \int (-T_0^0 + T_k^k)\sqrt{-g}\, d^3x\,, \tag{20}$$

where g is the determinant of the metric $g_{\mu\nu}$. This is the starting point of our analysis. We discuss this integral to find the mass formula of a quasi-black hole. For the mass formula for black holes, rather than quasi-black holes, see [18, 19, 20, 21], and, e.g., [22].

A. The various masses

1. *Total mass*

Since the spacetime is static by assumption, the distribution of matter does not depend on time. Then using (20) and noting from (3) that $\sqrt{-g} = N\sqrt{g_3}$, where g_3 is the determinant of the spatial part of the metric (3), i.e., is the determinant of the metric on the hypersurface $t =$ constant, one finds

$$M = \int (-T_0^0 + T_k^k)\, N\sqrt{g_3}\, d^3x\,. \tag{21}$$

The mass (21) can be split into three different contributions, from the inner region, the outer one (where, for example, a long-range electromagnetic field, or other matter fields, such as rings, can be present) and from the surface between the two,

$$M_{\rm tot} = M_{\rm in} + M_{\rm surf} + M_{\rm out}\,, \tag{22}$$

where, $M_{\rm in}$, $M_{\rm surf}$, $M_{\rm out}$, are the inner mass, the surface mass, and the outer mass, respectively. Let us study each mass term in turn.

2. *Inner mass*

From Eq. (21), the inner mass is given by the expression,

$$M_{\rm in} = \int_{\rm inner} (-T_0^0 + T_k^k)\, N\sqrt{g_3} d^3x \ . \tag{23}$$

As we presume a quasi-black hole to form, it means that in the entire inner region $N \leq N_{\rm B}$ where $N_{\rm B}$ is the maximum boundary value and, as $N_{\rm B} \to 0$, also $N \to 0$ everywhere in the inner region [1]. Therefore, one can write the following inequality for the inner mass, $M_{\rm in} = \int_{\rm inner} (-T_0^0 + T_k^k)\, N\sqrt{g_3}\, d^3x \leq N_{\rm B} \int (-T_0^0 + T_k^k)\sqrt{g_3}\, d^3x$. Defining the proper mass M_0 as $M_0 \equiv -\int T_0^0 \sqrt{g_3}\, d^3x$, and the mass due to the stresses as $M_k \equiv \int T_k^k \sqrt{g_3}\, d^3x$, one finds

$$M_{\rm in} \leq N_{\rm B}\,(M_0 + M_k)\,. \tag{24}$$

By assumption, the proper mass M_0 is finite. Assuming also that $T_k^k \leq C\,|T_0^0|$ where C is some constant, we obtain that M_k is finite as well. Thus, in the quasi-black hole limit,

$$M_{\rm in} = 0\,, \tag{25}$$

due to the factor $N_{\rm B}$.

3. *Surface mass*

Now consider the contribution from the surface,

$$M_{\rm surf} = \int_{\rm surface} (-T_0^0 + T_k^k)\, N\sqrt{g_3}\, d^3x \ . \tag{26}$$

Here, there are delta-like contributions in T_μ^ν, the surface stresses being then given by,

$$S_\mu^\nu = \int T_\mu^\nu\, dl\,, \tag{27}$$

where the integral is taken across the shell. Define α as,

$$\alpha = 8\pi (S_a^a - S_0^0)\,, \tag{28}$$

so that from a combination of some of the equations above, we get,

$$M_{\rm surf} = \frac{1}{8\pi} \int \alpha\, N\, d\sigma\,, \tag{29}$$

where $d\sigma = \sqrt{g_2}\, d^2x$, g_2 being the determinant of the metric spanned by the x^a (see Eq. (3)). Now, one also has the relationship $8\pi S_\mu^\nu = [[K_\mu^\nu]] - \delta_\mu^\nu[[K]]$, where K_μ^ν is the extrinsic curvature tensor, $[[...]] = [(...)_+ - (...)_-]$, and subscripts "+ " and "−" refer to the outer and inner sides, respectively (see, e.g., [13, 14]). Thus $\alpha = 8\pi(S_a^a - S_0^0) = -2[[K_0^0]]$. Put n^μ as the unit vector normal to the boundary surface. Then $K_0^0 = -\, n^0{}_{;0} = -\frac{1}{N}\frac{\partial N}{\partial l}$. As a result, we obtain

$$\alpha = \frac{2}{N}\left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] , \tag{30}$$

and so,

$$M_{\rm surf} = \frac{1}{4\pi}\int_{\rm surf}\left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] d\sigma\, . \tag{31}$$

This shows clearly that one cannot ignore the surface stresses in the non-extremal case. This is a very important feature of non-extremal configurations which can be confronted with the extremal ones. One could naïvely think that one could simply restrict oneself to the case of vanishing stresses but in the problem under discussion this is impossible. Indeed, it is seen that the stresses enter the mass formulas via the quantity α, so in the case of vanishing stresses $M_{\rm surf}$ would also vanish. But this is obviously impossible in the non-extremal case. Indeed, in the quasi-black hole limit, the situation we want to analyze in detail, one has $(\frac{\partial N}{\partial l})_- \to 0$, since everywhere in the inner region N is bounded and tends to zero by the definition of a quasi-black hole [1], so that $(\frac{\partial N}{\partial l})_- \to 0$. Thus, in the limit, $\alpha \to \frac{2}{N}(\frac{\partial N}{\partial l})_+$. Now, according to (11) and (12), one has $\alpha \simeq 2\frac{\kappa}{N}$, where κ is the surface gravity. It diverges since in general at the quasihorizon $\kappa \neq 0$ and $N \to 0$. But the surface contribution to the mass is finite due to the factor N. So,

$$M_{\rm surf} = \frac{\kappa A_{\rm h}}{4\pi}\, , \tag{32}$$

where $A_{\rm h}$ is the quasihorizon area. Eq. (32) is valid in general, i.e., it is valid in the non-extremal case where $\kappa \neq 0$, and in the extremal case where, since $\kappa = 0$, the surface mass is zero, $M_{\rm surf} = 0$.

Let us study the extremal case in more detail. In the extremal case one has, near the quasihorizon, $l \to \infty$ and $N \sim \exp(-Bl)$ where B is a constant. Therefore, $\alpha = -2B$ is finite (for instance, for a system forming a quasi-black hole whose exterior metric is Reissner-Nordström, one has $B = \frac{1}{r_+}$, where r_+ is the horizon radius, and so, $\alpha = -\frac{2}{r_+}$, finite). Then, it follows from (31) that $M_{\rm surf} = 0$. The surface stresses themselves are not equal to zero but

they do not contribute to the mass in the extremal case, since it is multiplied by the factor N in (29). For completeness, we also mention the ultraextremal case, with the asymptotic behavior of the lapse function being $N \sim l^{-s}$, $s>0$. Then, it is seen from (31) that not only the contribution of the surface to the total mass vanishes, but the surface stresses themselves vanish as well.

4. Outer mass

The outer mass is given by the expression,

$$M_{\rm out} = \int_{\rm out} (-T_0^0 + T_k^k)\, N\sqrt{g_3} d^3x \,. \tag{33}$$

Further, we may split $M_{\rm out}$ into an electromagnetic part $M_{\rm out}^{\rm em}$, and a non-electromagnetic part, $M_{\rm out}^{\rm matter}$ say, for the case of dirty black holes or dirty quasi-black holes, exactly in the manner as it was already done in [19], and obtain $M_{\rm out} = M_{\rm out}^{\rm em} + M_{\rm out}^{\rm matter}$. Since $M_{\rm out}^{\rm \ em} = \varphi_{\rm h} Q$, as explained below, where $\varphi_{\rm h}$ is the electric potential on the horizon in the case of black holes, and the electric potential on the quasihorizon in the case of quasi-black holes, and Q is the corresponding electric charge, one finds

$$M_{\rm out} = \varphi_{\rm h} Q + M_{\rm out}^{\rm matter} \,. \tag{34}$$

Now, we justify that $M_{\rm out}^{\rm em} = \varphi_{\rm h} Q$. As is explained above, the inner contribution of the matter to the mass vanishes (independently of the kind of matter or field), provided the components of the stress-energy tensor within the matter remain finite. The surface contribution has already been taken into account. Therefore, the only contribution of the electromagnetic field that survives in the quasihorizon limit is due to the outer region. As we will see for the issue under discussion the situation with quasi-black holes is very close to that with black holes. We only repeat briefly the main standard steps that lead to the corresponding expression. Consider the electromagnetic contribution to the external mass, $M_{\rm out}^{\rm em} = \int_{\rm out} (-T^{{\rm em}\,0}_{\ \ \ 0} + T^{{\rm em}\,k}_{\ \ \ k})\, N\sqrt{g_3}\, d^3x$. Using the expression for the electromagnetic field tensor $T^{{\rm em}\,\nu}_{\ \ \ \mu} = \frac{1}{4\pi}(F^\mu_\mu F^\mu_\nu - \frac{1}{4}\delta^\nu_\mu F_{\mu\nu}F^{\mu\nu})$, where $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$ is the field tensor, and A_μ is the four-potential, one can transform $M_{\rm out}^{\rm em}$ into $M_{\rm out}^{\rm em} = -\frac{1}{4\pi}\int_{\rm out} F_{0k}F^{0k}\, N\sqrt{g_3}\, d^3x$. The next step consists in an integration by parts, applying the Maxwell equation $\frac{\partial_\nu (F^{\mu\nu} N\sqrt{g_3})}{N\sqrt{g_3}} = 4\pi j^\mu$, where j^μ is the current, and the Gauss theorem. This operation converts $M_{\rm out}^{\rm em}$ into an

integral over a surface at infinity and a surface at the boundary of the quasihorizon. The first contribution vanishes since, by assumption, there are no currents at infinity. The second one gives us $M^{\rm em}_{\rm out} = A_0 = \varphi_{\rm h} Q,$ where Q is the charge enclosed within the quasihorizon, and φ_h is the electric potential on the horizon which is uniform in the quasihorizon limit (see below). Thus, although the expression $M^{\rm em}_{\rm out} = \varphi_{\rm h} Q$ comes from the outer contribution, it simply reduces to a surface term similarly to what happens in the black hole case [18, 19].

Now, we study in detail the behavior of the electric potential in the quasihorizon limit, and show that on the quasihorizon φ becomes constant. Indeed, in the derivation of the electromagnetic contribution to the mass, we used the uniformity of the electric potential in the quasihorizon limit, so that it can be pulled out of the surface integral, so now we have to prove it. The proof can be outlined in a way similar to the discussion of the surface gravity in Sec. II B. We require the finiteness of the electromagnetic energy density which is equal to $\rho^{\rm em} = -T^{{\rm em}\,0}_{\ \ \ \ 0} = \left(F_{0i} F_{0k} g^{ik}\right)/8\pi = \left(F^2_{0l} + g^{ab} \frac{\partial\varphi}{\partial x^a} \frac{\partial\varphi}{\partial x^b}\right) / \left(8\pi N^2\right)$. As the metric g^{ab} is positive definite, all terms enter this expression with the "+" sign, so that $N^{-1} \frac{\partial\varphi}{\partial x^a}$ should be finite. Near the quasi-black hole limit, it is equal to $N_0^{-1} \frac{\partial\varphi}{\partial x^a}$ where $N_0 = N_0(x^a)$ is the value of the lapse function on the quasihorizon. Then, taking the limit $N_0 \to 0$, we obtain that the finiteness of ρ entails that in the quasihorizon limit $\frac{\partial\varphi}{\partial x^a} \to 0$, so that φ indeed becomes constant and can be indeed pulled out from the surface integrals.

B. The mass formula

1. *The formula*

Putting all the masses together, the inner, the surface, and the outer masses, we find that the total mass of a system containing a quasi-blak hole is

$$M = \frac{\kappa A_{\rm h}}{4\pi} + \varphi_{\rm h} Q + M^{\rm matter}_{\rm out}\,. \qquad (35)$$

Note that for the extremal case, the term $\frac{\kappa A_{\rm h}}{4\pi}$ goes to zero, since κ is zero. Now, Eq. (35) is nothing else than the mass formula for quasi-black holes and surroundings, which has precisely the same form as the mass formula for black holes and surroundings ([18, 19, 20, 21, 22]). In outer vacuum, where $M^{\rm matter}_{\rm out} = 0$, one has the mass of the quasi-black hole is

$$M_{\rm h} = \frac{\kappa A_{\rm h}}{4\pi} + \varphi_{\rm h} Q\,. \qquad (36)$$

This is Smarr's formula [21] (see also [22]), but now for quasi-black holes. Note that, if one considers a generic matter configuration without a quasihorizon, the above arguments do not work at all. So Eqs. (35) and (36) are only valid for quasi-black holes, and black holes.

2. *Example: spherically symmetric electrically charged quasi-black hole*

Consider, as an example, the spherically symmetrical case when only the electromagnetic field is present outside, so that the outer region of the quasi-black hole is described by the Reissner-Nordström metric, i.e., $ds^2 = -\left(1 - 2m/r + Q^2/r^2\right) dt^2 + dr^2/\left(1 - 2m/r + Q^2/r^2\right) + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$. Then, $A_{\rm h} = 4\pi r_{\rm h}^2$, $\kappa_{\rm h} = (r_{\rm h} - m)/r_{\rm h}^2$, $\varphi(r_{\rm h}) = Q^2/r_{\rm h}$, and $r_{\rm h} = m + \sqrt{m^2 - Q^2}$. Using (35)-(36) one finds $M = M_{\rm h} = m$, as it should. This is valid in the non-extremal as well as in the extremal cases. The extremal case is peculiar, since for $m = Q = r_{\rm h}$, the surface contribution vanishes, and the contribution for the mass is purely electromagnetic. It is instructive to work out directly from Eqs (29) and (34). Then, $M_{\rm surf} = m - \;\; Q^2/r_{\rm h}$, $M_{\rm out} = Q^2/r_{\rm h}$, so that the total mass is equal to m. Clearly, in the particular case of an extremal quasi-black hole, again the surface contribution vanishes.

3. *Hairy properties of quasi-black holes: Mass, electric potential, and charge*

Now, it is interesting to understand which quantities give hair and which give no hair to the quasi-black holes. Let us start with the mass. The inner mass properties discussed above show that there are different quasi-black hole configurations characterized by the same mass but different inner mass densities $T_0^0 = \rho$ say. However, this difference becomes negligible in the quasi-black hole limit since ρ is multiplied by the factor N which, in this limit, vanishes in the inner region. This means that the hairy remnants of the original configuration, which exist in the mass density, are deleted in the quasihorizon limit. It is also instructive to look at the situation with the electromagnetic potential. From the definition of a quasi-black hole, it follows that in the inner region the lapse function N goes as $N = \varepsilon f(x^i)$, for some non-zero well-behaved $f(x^i)$, and with $\varepsilon \to 0$ in the quasi-black hole limit. Consider a static distribution of charge. Then the only non-zero component of the current j^μ is $j^0 = \frac{\rho_{\rm e}}{N\sqrt{g_3}}$, where $\rho_{\rm e}$ is the invariant electrical charge density. Then, it follows that $F^{0k} \sim \varepsilon^{-1}$ and so

$F_{k0} = \frac{\partial\varphi}{\partial x^k} \sim \varepsilon$. Therefore, the potential in the inner region takes the form

$$\varphi = \text{const} + \varepsilon h(x^i)\,, \tag{37}$$

for some h. Thus, φ tends to a constant in the quasihorizon limit (see also the discussion in [4] on hairy properties for particular spherically-symmetrical models). Again, as for the mass, the hairy remnants of the original inner configuration, which exist in the electric potential, are deleted in the quasihorizon limit. Finally, let us see what happens to the the charge distribution. The situation in this case somewhat different. The charge Q is defined by $Q = \int \rho_{\rm e}\sqrt{g_3}\, d^3x$, with no multiplication by N inside the integral. So different charge density distributions can be considered as yielding some kind of hair. However, to probe the corresponding details, an outer observer should exchange information with an inner observer. But, this is impossible because both the infinite tidal forces and the rescaling of time coordinate used by the two observers, do not allow such an exchange [1]. Therefore, even if there is hair, it appears in places that become unavailable for observations in the quasi-black hole limit. In this respect, the transition from a black hole to a quasi-black hole agrees with the no-hair theorems, thus extending their meaning.

4. Beyond the mass formula: corrections

In the discussion above, we have already pointed that quasi-black and black holes are distinct physical objects, and the method of derivation of the formulas for both objects is different. This difference is surely not revealed in the final mass formula (35) (or Eq. (36)). This, in a sense, is quite natural since for an outer observer the quasi-black hole is practically indistinguishable from a black hole (stressing again that in general this is not so for the inner region). However, the difference should be manifest in correction terms which reflect how close the system is to the quasi-black hole state. Therefore, it is of interest to evaluate these corrections which do not have an analogue for the black hole case. Besides, this evaluation shows the accuracy of the formula. To this end, we discuss the different contributions separately. For the inner and outer masses the answer is simple. On the other hand, the surface contribution to the mass and the term due to the charge and electric potential require a careful evaluation.

According to the property (b) of quasi-black holes, $N \leq N_{\rm B} \sim \varepsilon$ on the boundary near

the quasihorizon and in the inner region where one can write $N = \varepsilon f(x^i)$. Therefore, it follows from (23) directly that the correction $M^{(1)}_{\rm in}$ to the mass is given by, $M^{(1)}_{\rm in} = O(\varepsilon)$. For the outer mass there is no need in such an evaluation at all since there are no specific features for a quasi-black hole there, it simply is given by the contribution to the mass of the region outside the body's boundary.

Let us now find the correction to the mass surface contribution, by evaluating the main correction stemming from the term (31), responsible for the surface contribution. The "-" term is of the order ε as it follows from the form of N in the inner region, listed above. To evaluate the "+" term, we consider first the non-extremal case. In the zeroth approximation we can surely neglect the difference between a quasi-black and a black hole in the "+" term, and use the expansion $N = \kappa l + O(l^2)$, where l is the proper distance to the quasihorizon (or to the horizon in the black hole case). As a result, $\left(\frac{\partial N}{\partial l}\right)_+ = \kappa + O(l)$, where $l \sim N \sim \varepsilon$. The evaluation for the extremal and ultraextremal cases is similar. One only has to take into account that, in the extremal case in (31) one has $\left(\frac{\partial N}{\partial l}\right)_+ \sim N \sim \exp(-\frac{l}{l_0}) \sim \varepsilon$. In the ultraextremal case the corrections from the "+" side turn out to be smaller than ε, since in the limit $l \to \infty$, one has $\left(\frac{\partial N}{\partial l}\right) \sim \frac{N}{l} \sim l^{-n-1} \sim \varepsilon^{1+\frac{1}{n}} \ll N \sim \varepsilon$, for some exponent n.

The correction connected with the surface contribution of the electromagnetic field, $M^{(1)\rm em}_{\rm surf}$, is given by

$$M^{(1)\rm em}_{\rm surf} = \frac{1}{4\pi}\int d\sigma_{0k} F^{0k}(\varphi - \varphi_H) + \varphi_H \Delta q\,, \tag{38}$$

where $d\sigma_{0k} = n_k\, d\sigma$ is the standard surface element of a two-dimensional surface, n_k is the unit normal to the surface, $q = \frac{1}{4\pi}\int d\sigma_{0k}F^{0k}$, Δq is the charge enclosed between the quasihorizon (or of the horizon in the black hole case) and the boundary surface that approaches it, and $\varphi_H =$ constant is the value of the potential on the quasihorizon (or on the horizon in the black hole case, but here the difference between a black hole and quasi-black hole is negligible). Using Eq. (37), which is valid in some vicinity of the quasihorizon (or horizon in the black hole case) on both sides, we see that the first term in (38) is of the order ε. For the second term we can write, $\Delta q \sim \Delta A$, where ΔA is the difference between the surface areas. Then, $\Delta A \sim g^{ab}\Delta g_{ab}$. In the non-extremal case $\Delta g_{ab} \sim l \sim N \sim \varepsilon$. In the extremal case it follows from (18) that $\Delta g_{ab} \sim \exp(-\frac{l}{l_0}) \sim N \sim \varepsilon$. In the ultraextremal case, we have by definition $N \sim l^{-n}$ (see Sec. II B 2) and $\Delta g_{ab} \sim l^{-s}$, where $n > 0$ and $s > 0$ (a more detailed discussion of the properties of such metrics is contained in [8, 9, 10]). Therefore, $\Delta g_{ab} \sim \varepsilon^p$, with $p = \frac{s}{n}$, and depending on the relation between n and s, both $p > 1$ or $p < 1$

are possible. Then, $M^{(1)\mathrm{em}}_{\mathrm{surf}} = O(\varepsilon^p)$.

Thus, in brief, in the non-extremal and extremal cases all corrections are of the order ε, whereas in the ultraextremal case the several different corrections may contain ε in different powers as described above.

5. Other black hole mimickers: Gravastars

In this study of static spacetimes, we have mainly concentrated on comparing aspects of quasi-black holes to true black holes. But there are other interesting objects. Indeed, in recent years, there has been some debate to what extent observational data can favor the existence of black holes or can be ascribed to objects with size close to their own gravitational radius but having no horizon, the black hole mimickers [2]. Therefore, it is appropriate to compare properties, such as the mass formula in the present article, of quasi-black holes, not only to those of true black holes but also to other types of mimickers. In this connection, we make some short remarks about one of the most prominent mimickers, namely, gravastars [23]. Their distinctive feature consists in that the almost Schwarzschild-like outer metric is combined with a de Sitter-like inner one. Then, for our context, the difference between both types of mimickers, i.e., quasi-black holes and gravastars, reveals itself in the behavior of the lapse function in the inner region and on the boundary surface. For gravastars, in contrast to quasi-black holes, in the inner region N does not vanish and is a monotonically decreasing function of the radial coordinate. Therefore, it has a non-vanishing derivative $\left(\frac{\partial N}{\partial l}\right)_-$ on the boundary, which, in turn, affects the mass value according to Eq. (31). As a result, both the inner region and surface contribute to the mass, this contribution being model-dependent. For quasi-black holes, as we have been discussing, all the information about the inner region and boundary is deleted and the answer for the mass formula has a universal form, just in the same spirit of black hole physics. In this sense, quasi-black holes represent configurations which are closer to black holes than gravastars, indeed they are much better mimickers (although qualitative differences between quasi-black holes and true black holes persist anyway, see [1] and [2]).

IV. CONCLUSIONS: THE ABRAHAM-LORENTZ ELECTRON AND OTHER DISCUSSIONS

A. The Abraham-Lorentz electron and extremal quasi-black holes

The above results have a rather unexpected implication concerning another topic, namely, the problem of a self-consistent analogue of an elementary particle in general relativity having a mass of pure electromagnetic origin. This is the correspondent to the Abraham-Lorentz electron in flat spacetime. In flat spacetime Coulomb repulsion prevents such a construction, so one needs Poincaré stresses for such a construction, But, by including gravitation, one may possibly dispense with those stresses, the attractive force of gravitation making the question reasonable within the theory of general relativity. On a first glance, it would seem natural that, as we want to have electromagnetic and gravitational forces alone, we must require the absence of a bare tension on the surface. Otherwise, this would mean that apart from electromagnetism and gravitation, there were also external forces of different nature, of Poincaré type, needed to keep the system in equilibrium. An attempt of this kind was made on [24], where it was argued that a charged shell with empty space inside obeys this criteria in the extremal case, $M = Q$. This was criticized in [25] where it was shown that, actually, the surface stresses do not vanish in such a model even in the extremal limit. Instead, another model was suggested in [25], where the external extremal Reissner-Nordström metric was glued to the Bertotti-Robinson tube-like geometry inside. Then, it turned out that the surface stresses vanish in the limit when the surface of gluing approaches the horizon.

However, it follows from the results of the present article that the two issues "mass of pure electromagnetic origin" and "absence of bare stresses" in general relativity may be different in one exceptional situation. If the surface of the charged body approaches the quasihorizon, the contribution of the bare tension on the surface to the total mass in the extremal case completely vanishes, although these stresses by themselves remain finite. As a result, we obtain a model in which a distant observer measures a mass as having purely electromagnetic origin, although locally, on the surface there are extraneous additional forces. Moreover, one can even allow non-electromagnetic fields inside in the bulk, since anyway, their contribution to the total mass vanishes in the quasihorizon limit. All the region beyond the quasihorizon including the quasihorizon itself turns out to be frozen and gives no contribution to the mass

(for the non-extremal case the inner region also is frozen but the boundary is not).

It is also worth noting that the general statement of [24] about the distinguished role of extremal black holes (now we would rephrase it as "black and quasi-black holes") turns out to be correct. They are suitable candidates for the role of classical models of elementary particles since only in this case the mass can have pure electromagnetic origin. Thus, in summary, as a by-product, we have obtained that an extremal quasi-black hole (in contrast to the non-extremal one) can serve as a physically reasonable classical model of an Abraham-Lorentz electron in that both the inner and surface contribution of forces with non-electromagnetic origin vanish. In doing so, we showed that one may weaken the requirement of vanishing surface stresses since the finite stresses have zero contribution to the total mass.

B. Other discussions

We have traced how the limiting transition from the static configuration to the quasi-black hole state reveals itself in the mass formula. It turned out that there is a perfect one-to-one correspondence between the different contributions for the total mass of a quasi-black hole and the mass formula for black holes. In particular, the inner contribution to the total mass vanishes in the quasi-black hole limit, and surely it is absent in the black hole case from the very beginning. The contribution of the surface stresses in the quasi-black hole corresponds just to the contribution from the horizon surface of a black hole. This is non trivial, since the corresponding terms have quite different origins. In the quasi-black hole case they are due to the boundary between both sides of the surface. In the black hole case only one side, the external, is relevant and the integrand over this surface has nothing to do with the expression for surface stresses. Nonetheless, both terms coincide in the limit under discussion.

The essential difference between non-extremal and extremal quasi-black holes consists in that the first case the surface stresses become infinite but have finite contribution to the total mass, while in the second case they are finite but have zero contribution to the total mass. Actually, we extended in the present paper the notion of a quasi-black hole admitting infinite surface stresses. As far as the mass is concerned, in the non-extremal case the surface of a quasi-black hole reveals itself in a way similar to a membrane in the membrane

paradigm setup [26], whereas in the extremal one we have "membrane without membrane", paraphrasing famous Wheeler's remarks [27]. By itself, the system with infinite stresses looks unphysical and this was the reason why non-extremal quasi-black holes were rejected in [1]. Nonetheless, consideration of such systems has at least methodical interest since it helps to understand better the relationship between quasi-black holes and black holes and the distinction between non-extremal and extremal limits in this context. In particular, it is of interest to trace the similarity and distinction between quasi-black holes and black holes from the viewpoint of the membrane paradigm in a more general setting.

Of course, by adding rotation all these matters may become even more interesting.

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